On embedding of classes H_p^ω

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Introduction

Let $\omega(\delta)$ be a nondecreasing continuous function on the interval $(0, 1)$ having the properties:

$$\omega(0) = 0, \quad \omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2) \quad \text{for} \quad 0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 1.$$

Such a function will be called a "modulus of continuity".

If $f(x) \in L^p(0, 1)$ $(1 \leqq p < \infty)$, the "modulus of continuity of $f(x)$ in $L^p(0, 1)$" is defined by

$$\omega_p(\delta, f) = \sup_{0 \leqq h \leqq \delta} \left\{ \int_0^{1-h} |f(x+h)-f(x)|^p \, dx \right\}^{1/p} \qquad (0 \leqq \delta \leqq 1).$$

If given a number $p \geqq 1$ and a modulus of continuity $\omega(\delta)$, then $H_p^\omega \equiv H_p^{\omega(\delta)}$ will denote the collection of the functions $f(x)$ satisfying the condition $\omega_p(\delta, f) = O(\omega(\delta))$.

Let $\varphi(x)$ be a nonnegative nondecreasing function on $[0, \infty)$. The collection of the functions $f(x)$ having the property

$$\int_0^1 |f(x)|^r \varphi(|f(x)|) \, dx < \infty$$

will be denoted by $L^r \varphi(L)$.

Recently P. L. UL'JANOV has investigated in several papers (see for instance [4], [5] and [6]) the following problems:

1) Find a sufficient condition that

$$f \in L^r \varphi(L).$$

2) Find necessary and sufficient conditions in order that

$$H_p^{\omega(\delta)} \subset L^r \varphi(L).$$

*) This research was made while the author worked in the Steklov Institute Moscow as a visiting scientist.

Among others he proved the following theorems:

Theorem A. ([5], Theorem 1) *Suppose that* $f(x)\in L^p(0,1)$ *for some* $p\geqq 1$. *If* $p=1$ *then*

a) $$\sum_{n=1}^{\infty}[\varphi(n+1)-\varphi(n)]\,\omega_1\left(\frac{1}{n},f\right)<\infty \quad \textit{implies} \quad f\in L\varphi(L),$$

b) $$\sum_{n=1}^{\infty}\frac{1}{n^2}\,\varphi\left(36n\omega_1\left(\frac{1}{n},f\right)\right)<\infty \quad \textit{implies} \quad f\in\varphi(L),$$

while if $\nu\geqq p\geqq 1$ *then*

c) $$\sum_{n=1}^{\infty} n^{\frac{\nu}{p}-2}\,\omega_p^{\nu}\left(\frac{1}{n},f\right)<\infty \quad \textit{implies} \quad f\in L^{\nu}.$$

Theorem B. ([6], Theorem 3) *Suppose that* $1\leqq p<\nu<\infty$ *and* $0\leqq\beta<\infty$. *Then a necessary and sufficient condition for*

$$H_p^{\omega(\delta)}\subset L^{\nu}(\ln^+L)^{\beta}$$

is that

$$\sum_{n=1}^{\infty} n^{\frac{\nu}{p}-2}\,\omega^{\nu}\left(\frac{1}{n}\right)\ln^{\beta}(n+1)<\infty.$$

Theorem C. ([5], Theorem 2 and 4) *Let* $p=1$ *or* $p=2$. *Then a necessary and sufficient condition for*

$$H_p^{\omega(\delta)}\subset L^p\ln^+L$$

is that

$$\sum_{n=1}^{\infty}\frac{\omega^p(1/n)}{n}<\infty.$$

We remark that the method of proof of Theorem C for $p=1$ is different from what is given for $p=2$; furthermore, for $p=1$, Ul'janov proved the following more general theorem:

Theorem D. ([5], Theorem 2) *If* $\varphi(t)$ *is an even and nonnegative function such that*

$$\varphi(t)\uparrow\infty \quad \textit{as} \quad t\uparrow \quad (0\leqq t<\infty), \qquad \varphi(n+1)-\varphi(n)\downarrow \quad \textit{as} \quad n\uparrow\infty,$$

and

$$\varphi(t^2)\leqq\varphi(t) \quad \textit{for any} \quad 0\leqq t<\infty,$$

then, for a given modulus of continuity $\omega(\delta)$,

$$H_1^{\varphi(\delta)}\subset L\varphi(L)$$

if and only if

$$\sum_{n=1}^{\infty}(\varphi(n+1)-\varphi(n))\,\omega\left(\frac{1}{n}\right)<\infty.$$

In connection with Theorem C, P. L. UL'JANOV raised in a conversation the problem: Is Theorem C valid for any $p \geqq 1$?

In the present paper we are going to give an affirmative answer to this question; that is, we prove that Theorem C can be extended to any $p \geqq 1$, moreover we generalize all the above mentioned theorems. We would like to point out that the most important parts of our theorems are the cases $p = \nu > 1$.

Our method of proof is partly similar to that of UL'JANOV's but in the case $p = \nu > 1$ it is quite different from his. In this case ($p = \nu > 1$) the kernel of our proof can be found in Lemma 7 and Lemma 8.

Theorem 1. *Let $\{\varphi_k\}$ be a nonnegative monotonic sequence of numbers, $\nu \geqq 1$ and $f(x) \in L^p(0, 1)$ for some $p \geqq 1$. Define*

$$\Phi(x) = \sum_{k=1}^{x} k^{\frac{\nu}{p}-2} \varphi_k.\ ^{1)}$$

Then

(1) $$f \in L^{\nu - \frac{\nu}{p} + 1} \Phi(L)$$

follows from

(2) $$\sum_{k=1}^{\infty} k^{\frac{\nu}{p}-2} \varphi_k \omega_p^\nu \left(\frac{1}{n}, f\right) < \infty$$

in the following cases:

a) *if $\nu = p = 1$;*

b) *if $\nu = p > 1$ and*

(3) $$\sum_{k=m}^{\infty} \frac{\varphi_k}{k^{1+\varepsilon}} \leqq K \frac{\varphi_m}{m^\varepsilon}\ ^{2)}$$

for certain $\varepsilon = \varepsilon(p) > 0$[3]*;*

c) *if $\nu > p \geqq 1$ and*

(4) $$\varphi_k \leqq \varphi_{k+1}, \quad \sum_{k=m}^{\infty} \frac{\varphi_k}{k^2} \leqq K \frac{\varphi_m}{m}.$$

This theorem has the following corollary which was proved only for $p = 1$ till now (see Theorem A).

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[1]) Σ_a^b, where a and b are not integers, means a sum over all integers between a and b.

[2]) K and K_i denote either absolute constants or constants depending on certain functions and numbers which are not necessary to explain in detail, not necessarily the same at each occurrence.

[3]) See more on the meaning of $\varepsilon(p)$ in the proof of Lemma 8.

Corollary 1. *If $p \geqq 1$, $\beta > -1$ and*

$$\sum_{n=2}^{\infty} \frac{1}{n} (\ln n)^{\beta} \omega_p^p \left(\frac{1}{n}, f\right) < \infty,$$

then

$$f \in L^p (\ln^+ L)^{\beta+1};$$

if, furthermore, $\gamma > -1$ and

$$\sum_{n=8}^{\infty} \frac{(\ln \ln n)^{\gamma}}{n \ln n} \omega_p^p \left(\frac{1}{n}, f\right) < \infty,$$

then

$$f \in L^p (\ln^+ \ln L)^{\gamma+1}.$$

Let $\psi(x)$ be a nonnegative increasing function having the properties:

(5) $$\frac{\psi(x)}{x} \uparrow \quad \text{and} \quad \frac{\psi(x)}{x^{\eta}} \downarrow \quad \text{for some} \quad \eta > 1 \quad \text{as} \quad x \uparrow \infty.$$

MULHOLLAND [3] investigated such functions and proved: *If $\psi(x)$ satisfies* (5), $\gamma > 1$ *and $\lambda_n \geqq 0$, then*

(6) $$\sum_{n=1}^{\infty} n^{-\gamma} \psi(\lambda_1 + \lambda_2 + \cdots + \lambda_n) \leqq K \sum_{n=1}^{\infty} n^{-\gamma} \psi(n\lambda_n)$$

holds, where $K = K(\psi, \gamma)$.

Using (6) and an estimate of UL'JANOV (Lemma 4) we can prove

Theorem 2. *If $\psi(x)$ satisfies* (5), $p \geqq 1$, *and*

(7) $$\sum_{n=1}^{\infty} \frac{1}{n^2} \psi\left(n^{1/p} \omega_p \left(\frac{1}{n}, f\right)\right) < \infty,$$

then

$$f \in \psi(L).$$

This is an analogue of the part b) of Theorem A for any $p \geqq 1$.

Finally we prove the following

Theorem 3. *Let $1 \leqq p \leqq \nu < \infty$. Let $\omega(\delta)$ be a given modulus of continuity and $\{\varphi_k\}$ be a nonnegative monotonic sequence of numbers satisfying $\varphi_{k^2} \leqq K\varphi_k$ for any k, and if $\nu > p$, then moreover let $\varphi_k \leqq \varphi_{k+1}$. Then a necessary and sufficient condition that*

(8) $$H_p^{\omega(\delta)} \subset L^{\nu - \frac{\nu}{p} + 1} \Phi(L)$$

is that

(9) $$\sum_{n=1}^{\infty} n^{\frac{\nu}{p} - 2} \varphi_n \omega^{\nu} \left(\frac{1}{n}\right) < \infty,$$

where $\Phi(x)$ means the same as in Theorem 1.

We remark that in the case $\nu > p$ and $\varphi_k = (\ln k)^\beta$ $(\beta \geqq 0)$ Theorem 3 includes Theorem B.

Corollary 2. *Let $p \geqq 1$ and $\beta > -1$. Then a necessary and sufficient condition for*

$$H_p^{\omega(\delta)} \subset L^p (\ln^+ L)^{\beta+1}$$

is that

$$\sum_{n=2}^{\infty} \frac{(\ln n)^\beta}{n} \omega^p \left(\frac{1}{n}\right) < \infty.$$

This corollary is a generalization of Theorem C for any $p \geqq 1$ and gives an answer to the problem of UL'JANOV mentioned above. However, for $p = 1$ this corollary and also the following Corollary 3 were proved by UL'JANOV ([5], Corollary 4).

Corollary 3. *Let $p \geqq 1$ and $\gamma > -1$. Then a necessary and sufficient condition for*

$$H_p^{\omega(\delta)} \subset L^p (\ln^+ \ln^+ L)^{\gamma+1}$$

is that

$$\sum_{n=10}^{\infty} \frac{(\ln \ln n)^\gamma}{n \ln n} \omega^p \left(\frac{1}{n}\right) < \infty.$$

My grateful acknowledgement is due to Professor P. L. UL'JANOV for having called my attention to this problem.

§ 1. Lemmas

We require the following lemmas.

Lemma 1. ([2], Lemma 3) *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of nonnegative numbers such that*

$$\sum_{k=n}^{\infty} \alpha_k = \eta_n \alpha_n.$$

Then for any $\gamma \geqq 1$

$$\sum_{k=1}^{\infty} \alpha_k \left(\sum_{n=1}^{k} \beta_n\right)^\gamma \leqq K \sum_{k=1}^{\infty} \alpha_k (\eta_k \beta_k)^\gamma.$$

Lemma 2. ([5], Lemma 4) *If $f(x) \in L^p(0, 1)$, $1 \leqq p < \infty$ and*

$$\psi_n(x) = n \int_{k/n}^{(k+1)/n} f(t)\,dt \quad \text{for} \quad x \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \qquad (k = 0, 1, \ldots, (n-1)),$$

then

$$\left\{\int_0^1 |f(x) - \psi_n(x)|^p\,dx\right\}^{1/p} \leqq 4\omega_p\left(\frac{1}{n}, f\right) \qquad (n = 1, 2, \ldots).$$

Lemma 3. ([5], Lemma 13) *Let $A(u)$ be a nonnegative nondecreasing function on $[0, \infty)$ such that $A(u^2) \leqq KA(u)$ for any $u \in [0, \infty)$ and let $B(u)$ be a nonnegative function on $(0, 1]$. Then*

$$\int_0^1 B(u) A(B(u))\, du < \infty \tag{1.1}$$

implies

$$\int_0^1 B(u) A\left(\frac{1}{u}\right) du < \infty. \tag{1.2}$$

Lemma 4. *Let $p \geqq 1$ and $f(x) \in L^p(0, 1)$. Let $F(z)$ be a nonnegative nonincreasing function such that*

$$\operatorname{mes}\{x\colon x \in [0, 1],\ |f(x)| > y\} = \operatorname{mes}\{z\colon z \in [0, 1],\ F(z) > y\}.$$

Then

$$\int_0^{1/n} F(z)\, dz \leqq K\omega_1\left(\frac{1}{n}, f\right) \tag{1.3}$$

and for $p > 1$

$$F(2^{-n-1}) \leqq K\left(1 + \sum_{k=1}^{n} 2^{k/p} \omega_p(2^{-k}, f)\right) \qquad (n = 1, 2, \ldots) \tag{1.4}$$

hold.

This lemma can be found in the proof of Theorem A ([5], Theorem 1) implicitly.

Lemma 5. ([5], Lemma 7) *If $f(x) \in L(0, 1)$ and $0 \leqq \alpha \leqq 1$, then*

$$\sup_{\substack{E \subset [0,1] \\ |E| = \alpha}} \int_E |f(x)|\, dx = \int_0^\alpha F(z)\, dz,$$

where $F(z)$ has the same meaning as in Lemma 4.

Lemma 6. ([1], p. 78) *If $\omega(\delta)$ is a modulus of continuity, then there exists a concave function $\omega_1(\delta)$ such that*

$$\omega(\delta) \leqq \omega_1(\delta) \leqq 2\omega(\delta) \quad \textit{if} \quad 0 \leqq \delta \leqq 1.$$

Lemma 7. *If $p \geqq 1$ and $f(x) \in L^p(0, 1)$, then*

$$\int_0^{1/2n} F(z)^p\, dz \leqq K(p)\left(\int_{1/2n}^{1/n} F(z)^p\, dz + \omega_p^p\left(\frac{1}{n}, f\right)\right) \tag{1.5}$$

for any $n \geqq 1$, where $K(p)$ depends only on p, and $F(z)$ means the same as in Lemma 4.

Proof. Let us choose $M(>1)$ such that

$$q=\left(1+\frac{1}{M}\right)^p<2. \tag{1.6}$$

Set

$$E_n^*=\left\{x\colon x\in[0,1],\ |f(x)|>F\left(\frac{1}{2n}\right)\right\}\quad\text{and}\quad E_n^{**}=\left\{x\colon x\in[0,1],\ |f(x)|=F\left(\frac{1}{2^n}\right)\right\}.$$

If $\operatorname{mes} E_n^*<\frac{1}{2n}$, then let us define y such that

$$\operatorname{mes}(E_n^{**}\cap(0,y))=\frac{1}{2n}-\operatorname{mes} E^*$$

and let

$$E_n=(E_n^{**}\cap(0,y))\cup E_n^*.$$

If $\operatorname{mes} E_n^*=\frac{1}{2n}$, then we set $E_n=E_n^*$.

First we estimate $|f(x)|^p$ on E_n. Let $\psi_n(x)$ be the same function as in Lemma 2. Since the estimate

$$|f(x)|^p\leqq 2^p(|f(x)-\psi_n(x)|^p+|\psi_n(x)|^p)$$

in not fine enough for our aim we split the set E_n into the parts

$$E_n'=\{x\colon x\in E_n,\ M|f(x)-\psi_n(x)\leqq|\psi_n(x)|\}$$

and

$$E_n''=\{x\colon x\in E_n,\ M|f(x)-\psi_n(x)|>|\psi_n(x)|\}.$$

Then we have by $|f(x)|\leqq|f(x)-\psi_n(x)|+|\psi_n(x)|$

$$\int_{E_n'}|f(x)|^p\,dx\leqq\int_{E_n'}\left(\frac{1}{M}+1\right)^p|\psi_n(x)|^p\,dx\leqq q\int_{E_n}|\psi_n(x)|^p\,dx$$

and

$$\int_{E_n''}|f(x)|^p\,dx\leqq\int_{E_n''}(M+1)^p|f(x)-\psi_n(x)|^p\,dx\leqq(M+1)^p\int_0^1|f(x)-\psi_n(x)|^p\,dx.$$

Hence we get by Lemma 2 and Lemma 5

$$\int_0^{1/2n}F(z)^p\,dz=\int_{E_n}|f(x)|^p\,dx\leqq q\int_{E_n}|\psi_n(x)|^p\,dx+K_1(p)\,\omega_p^p\left(\frac{1}{n},f\right)\leqq \tag{1.7}$$

$$\leqq q(\max|\psi_n(x)|)^p\frac{1}{2n}+K_1(p)\,\omega_p^p\left(\frac{1}{n},f\right).$$

Since

$$\max|\psi_n(x)|\leqq n\int_0^{1/n}F(z)\,dz\leqq n\left\{\int_0^{1/n}F(z)^p\,dz\right\}^{1/p}n^{\frac{1-p}{p}}\leqq\left\{n\int_0^{1/n}F(z)^p\,dz\right\}^{1/p},$$

we obtain by (1. 7)

$$\int_0^{1/2n} F(z)^p\,dz \leqq \frac{q}{2}\int_0^{1/n} F(z)^p\,dz + K_1(p)\,\omega_p^p\left(\frac{1}{n}, f\right).$$

Hence, by (1. 6),

$$\left(1-\frac{q}{2}\right)\int_0^{1/2n} F(z)^p\,dz \leqq \frac{q}{2}\int_{1/2n}^{1/n} F(z)^p\,dz + K_1(p)\,\omega_p^p\left(\frac{1}{n}, f\right)$$

which gives the conclusion (1. 5).

Lemma 8. *If $p \geqq 1$ and $f(x) \in L^p(0,1)$, then there exist constants $K(p)$ and $\varepsilon(p) > 0$ depending only on p such that*

$$\int_0^{1/n} F(z)^p\,dz \leqq \frac{K(p)}{n^{\varepsilon(p)}}\left(\sum_{k=1}^{n} k^{\varepsilon(p)-1}\,\omega_p^p\left(\frac{1}{k}, f\right) + \int_0^1 F(z)^p\,dz\right) \tag{1. 8}$$

for any $n \geqq 1$.

Proof. By Lemma 7 there exists an integer $N = N(p)$ such that for any $n \geqq 1$

$$\int_0^{2^{-n}} F(z)^p\,dz \leqq N\left(\int_{2^{-n-1}}^{2^{-n}} F(z)^p\,dz + \omega_p^p(2^{-n}, f)\right). \tag{1. 9}$$

Let us define for every $n \geqq 1$

$$a_n = \int_{2^{-n-1}}^{2^{-n}} F(z)^p\,dz, \quad b_n = \omega_p^p(2^{-n}, f)$$

and

$$\alpha_n = \sum_{k=(n-1)N+1}^{nN} a_k, \quad \beta_n = \sum_{k=(n-1)N+1}^{nN} b_k.$$

Considering (1. 9) we have

$$\sum_{k=n}^{\infty} a_k \leqq N(a_n + b_n)$$

and hence an easy computation gives that for any $m \geqq 1$

$$\sum_{i=m+1}^{\infty} \alpha_i \leqq \alpha_m + \beta_m.$$

Indeed, we have for any nonnegative integer j

$$\sum_{i=m+1}^{\infty} \alpha_i = \sum_{k=mN+1}^{\infty} a_k \leqq \sum_{k=mN+1-j}^{\infty} a_k \leqq N(a_{mN+1-j} + b_{mN+1-j}).$$

Taking $j=1, 2, \ldots, N$ and summing we obtain

$$N \sum_{i=m+1}^{\infty} \alpha_i \leqq N \sum_{k=(m-1)N+1}^{mN} (a_k+b_k) = N(\alpha_m+\beta_m),$$

and hence we get the required inequality. Multiplying this inequality by max $(2^{m-2}, 1)$ for all m, $1 \leqq m \leqq n$, summing and cancelling we obtain

$$2^{n-1} \sum_{i=n+1}^{\infty} \alpha_i \leqq \alpha_1+\beta_1+\sum_{k=2}^{n} 2^{k-2}\beta_k. \tag{1.10}$$

Inserting the definition of α_i and β_i, (1.10) implies

$$\int_0^{2^{-(nN+1)}} F(z)^p\,dz \leqq 2^{4-n}\left\{\int_0^1 F(z)^p\,dz+\sum_{k=1}^{n} 2^k N\omega_p^p(2^{-(k-1)N}, f)\right\} \leqq \tag{1.11}$$

$$\leqq \frac{K_1(p)}{2^n}\left\{\int_0^1 F(z)^p\,dz+\sum_{k=1}^{n} 2^k\omega_p^p(2^{-kN}, f)\right\}.$$

If $2^{nN+1} \leqq m < 2^{(n+1)N+1}$, then from (1.11) it follows with $\varepsilon = \frac{1}{N}$ that

$$\int_0^{1/m} F(z)^p\,dz \leqq \frac{K_2(p)}{m^\varepsilon}\left\{\int_0^1 F(z)^p\,dz+\sum_{k=1}^{\varepsilon \overset{2}{\log} m} 2^k\omega_p^p(2^{-kN}, f)\right\}. \tag{1.12}$$

From this point on the proof is an easy computation. Indeed, we have

$$\sum_{k=1}^{\varepsilon \overset{2}{\log} m} 2^k\omega_p^p(2^{-kN}, f) \leqq 2^N \sum_{k=1}^{\varepsilon \overset{2}{\log} m} \frac{2^k}{2^{kN}} \sum_{i=2^{(k-1)N}+1}^{2^{kN}} \omega_p^p\left(\frac{1}{i}, f\right) \leqq$$

$$\leqq 4^N \sum_{k=1}^{\varepsilon \log m} \sum_{i=2^{(k-1)N}+1}^{2^{kN}} \frac{i^\varepsilon}{i}\,\omega_p^p\left(\frac{1}{i}, f\right) \leqq 4^N \sum_{k=1}^{m} i^{\varepsilon-1}\omega_p^p\left(\frac{1}{i}, f\right);$$

inserting this into (1.12) we obtain (1.8) in accordance with our statement.

The following two lemmas are slight improvements of Lemmas 11 and 12 of Ul'janov [5].

Lemma 9. *Let $p>0$, $\alpha > 1-p$ and let $\omega(\delta)$ be a concave modulus of continuity. If the sequence $\{\varphi_k\}$ is monotonic,*

$$\varphi_k \geqq 0, \quad \sum_{k=m}^{\infty} \frac{\varphi_k}{k^{\alpha+p}} \leqq K\frac{\varphi_m}{m^{\alpha+p-1}} \tag{1.13}$$

and

$$\sum_{k=1}^{\infty} \varphi_k k^{-\alpha}\omega^p\left(\frac{1}{k}\right) = \infty, \tag{1.14}$$

then there exists a sequence of numbers $\{B_k\}$ *such that*

$$(1.15)\qquad B_k \downarrow 0, \quad B_k \leqq \omega\left(\frac{1}{k}\right), \quad \sum_{k=1}^{m} k^{p-1} B_k^p \leqq K m^p \omega^p\left(\frac{1}{m}\right)$$

and

$$(1.16)\qquad \sum_{k=1}^{\infty} \varphi_k k^{-\alpha} B_k^p = \infty.$$

Proof. In view of the conditions,

$$(1.17)\qquad n\omega\left(\frac{1}{n}\right) \uparrow \infty \quad \text{as} \quad n \to \infty.$$

By (1. 17) we can easily define a sequence of integers $\{n_k\}$ such that

$$(1.18)\qquad n_{k+1}\omega\left(\frac{1}{n_{k+1}}\right) > 2n_k\omega\left(\frac{1}{n_k}\right) \quad \text{for} \quad k \geqq 1$$

and

$$(1.19)\qquad n\omega\left(\frac{1}{n}\right) \leqq 2n_k\omega\left(\frac{1}{n_k}\right) \quad \text{if} \quad n_k \leqq n < n_{k+1}.$$

Indeed, let $n_0=0$ and $n_1=1$. Suppose that $n_1<n_2<\dots<n_k$ are defined so that they satisfy (1. 18) and (1. 19), and let n_{k+1} be the smallest integer μ having the property $\mu\omega\left(\frac{1}{\mu}\right)>2n_k\omega\left(\frac{1}{n_k}\right)$. Thus, by induction, we get the required sequence $\{n_k\}$.

Now we define the sequence $\{B_n\}$. Put

$$(1.20)\qquad B_n = \omega\left(\frac{1}{n_k}\right) \quad \text{for} \quad n_{k-1} < n \leqq n_k \qquad (k = 1, 2, \dots).$$

It is clear that $B_n \downarrow 0$ and $B_n \leqq \omega\left(\frac{1}{n}\right)$. To prove the estimation (1. 15) we choose r such that $n_r<m\leqq n_{r+1}$. Then, by (1. 18) and (1. 20), we have

$$\sum_{n=1}^{m} n^{p-1} B_n^p \leqq \sum_{k=1}^{r} \sum_{n=n_{k-1}+1}^{n_k} n^{p-1} B_n^p + \sum_{n=n_r+1}^{m} n^{p-1} B_n^p \leqq$$

$$\leqq K(p)\left(\sum_{k=1}^{r} B_{n_k}^p n_k^p + m^p B_m^p\right) \leqq K_1(p) m^p \omega^p\left(\frac{1}{m}\right)$$

in accordance with our statement.

Finally we prove (1. 16). Since $n_{k+1}>2n_k$ and $\varphi_{2i}\leqq K_1\varphi_i$ follow from (1. 13), we have

$$(1.21)\qquad \sum_{n=1}^{\infty} n^{-\alpha}\varphi_n B_n^p = \sum_{k=1}^{\infty}\sum_{n=n_{k-1}+1}^{n_k} n^{-\alpha}\varphi_n B_n^p \geqq$$

$$\geqq \sum_{k=1}^{\infty} B_{n_k}^p \cdot \sum_{n=\frac{n_k}{2}+1}^{n_k} \varphi_n n^{-\alpha} \geqq K_2 \sum_{k=1}^{\infty} B_{n_k}^p \varphi_{n_k} n_k^{1-\alpha}.$$

On the other hand by (1. 13) and (1. 19),

$$(1.22)\qquad \sum_{n=n_k}^{n_{k+1}-1} \varphi_n n^{-\alpha}\omega^p\left(\frac{1}{n}\right) = \sum_{n=n_k}^{n_{k+1}-1} \varphi_n n^{-\alpha-p}\left(n\omega\left(\frac{1}{n}\right)\right)^p \leqq$$

$$\leqq 2^p\left(n_k\omega\left(\frac{1}{n_k}\right)\right)^p \sum_{n=n_k}^{\infty}\varphi_n n^{-\alpha-p} \leqq K_3\left(\omega\left(\frac{1}{n_k}\right)\right)^p \varphi_{n_k} n_k^{1-\alpha} = K_3 B_{n_k}^p \varphi_{n_k} n_k^{1-\alpha}.$$

In view of (1. 14), (1. 21) and (1. 22) we obtain (1. 16).

The proof is thus completed.

Lemma 10. *Let* $\nu\geqq 1$, $1-\nu<\alpha<1$ *and let* $\omega(\delta)$ *be a concave modulus of continuity. If the positive sequence* $\{\varphi_k\}$ *is increasing,*

$$(1.23)\qquad \sum_{n=m}^{\infty} \varphi_n n^{-\alpha-\nu} \leqq K\varphi_m m^{1-\alpha-\nu}$$

and

$$(1.24)\qquad \sum_{n=1}^{\infty} \varphi_n n^{-\alpha}\omega^\nu\left(\frac{1}{n}\right) = \infty,$$

then there exist a sequence of numbers $\{B_n\}$ *and a sequence of integers* $\{n_k\}$ *such that*

$$(1.25)\qquad B_n\downarrow 0 \quad \text{and} \quad B_n\leqq \omega\left(\frac{1}{n}\right),$$

$$(1.26)\qquad n_{k+1}>2n_k \quad \text{and} \quad B_{n_{k+1}}\leqq \frac{1}{2}B_{n_k} \qquad (k\geqq 1),$$

$$(1.27)\qquad \sum_{n=1}^{m} n^{p-1}B_n^p \leqq K(p)m^p\omega^p\left(\frac{1}{m}\right) \quad \text{for any} \quad p>0,$$

$$(1.28)\qquad \sum_{n=1}^{\infty}\varphi_n n^{-\alpha}B_n^\nu = \infty \quad \text{and} \quad \sum_{k=1}^{\infty}\varphi_{n_k} n_k^{1-\alpha}B_{n_k}^\nu = \infty$$

and

$$(1.29)\qquad \sum_{n=1}^{\infty}\varphi_{2^n}2^{n(1-\alpha)}(B_{2^n}-B_{2^{n+1}})^\nu = \infty.$$

Proof. Since $\omega(\delta)$ is concave, by (1.23) we have $n\omega\left(\frac{1}{n}\right)\uparrow\infty$. Using this we can define the sequence $\{n_k\}$. Let $n_0=0$ and $n_1=1$. Suppose that $n_1<n_2<\cdots<n_k$ are defined so that $n_{i+1}>2n_i$ $(i=0, 1, \ldots, k-1)$. Then denote by m_{k+1} the smallest integer μ having the property $\mu\omega\left(\frac{1}{\mu}\right)>2n_k\omega\left(\frac{1}{n_k}\right)$. By the definition of m_{k+1} we have

$$(1.30)\qquad n\omega\left(\frac{1}{n}\right)\leqq 2n_k\omega\left(\frac{1}{n_k}\right)\quad\text{if}\quad n_k\leqq n<m_{k+1},$$

$$(1.31)\qquad m_{k+1}\omega\left(\frac{1}{m_{k+1}}\right)>2n_k\omega\left(\frac{1}{n_k}\right)$$

and

$$(1.32)\qquad m_{k+1}>2n_k.$$

If

$$(1.33)\qquad \omega\left(\frac{1}{m_{k+1}}\right)\leqq\frac{1}{2}\omega\left(\frac{1}{n_k}\right),$$

then set $n_{k+1}=m_{k+1}$. Conversely, if $\omega\left(\frac{1}{m_{k+1}}\right)>\frac{1}{2}\omega\left(\frac{1}{n_k}\right)$, then let n_{k+1} be the smallest integer μ satisfying $\omega\left(\frac{1}{\mu}\right)\leqq\frac{1}{2}\omega\left(\frac{1}{n_k}\right)$. In this case

$$(1.34)\qquad \omega\left(\frac{1}{n}\right)>\frac{1}{2}\omega\left(\frac{1}{n_k}\right)\quad\text{for}\quad n_k\leqq n<n_{k+1}$$

and

$$(1.35)\qquad \omega\left(\frac{1}{n_{k+1}}\right)\leqq\frac{1}{2}\omega\left(\frac{1}{n_k}\right).$$

Set

$$(1.36)\qquad B_n=\omega\left(\frac{1}{n_k}\right)\quad\text{for}\quad n_{k-1}<n\leqq n_k\qquad(k=1, 2, \ldots).$$

The statements (1.25) and (1.26) obviously follow from the definitions and (1.27) can be proved as in Lemma 9.

Now we prove (1.28). As in Lemma 9 we have

$$(1.37)\qquad \sum_{n=1}^{\infty}n^{-\alpha}\varphi_n B_n^{\nu}\geqq K\sum_{k=1}^{\infty}B_{n_k}^{\nu}\varphi_{n_k}n_k^{1-\alpha}.$$

If $n_{k+1}=m_{k+1}$, then by (1.23) and (1.30) we get

$$(1.38)\qquad \sigma_k\equiv\sum_{n=n_k}^{n_{k+1}-1}\varphi_n n^{-\alpha}\omega^{\nu}\left(\frac{1}{n}\right)=\sum_{n=n_k}^{n_{k+1}-1}\varphi_n n^{-\alpha-\nu}\left(n\omega\left(\frac{1}{n}\right)\right)^{\nu}\leqq$$
$$\leqq 2^{\nu}\sum_{n=n_k}^{\infty}\left(n_k\omega\left(\frac{1}{n_k}\right)\right)^{\nu}\varphi_n n^{-\alpha-\nu}\leqq K(\nu)\omega^{\nu}\left(\frac{1}{n_k}\right)\varphi_{n_k}n_k^{1-\alpha}.$$

If $n_{k+1} > m_{k+1}$, then by (1. 34) we obtain

$$(1.39)\qquad \sigma_k \leqq \omega^\nu\left(\frac{1}{n_k}\right)\sum_{n=n_k}^{n_{k+1}-1}\varphi_n n^{-\alpha} \leqq K_1(\nu)\,\omega^\nu\left(\frac{1}{n_{k+1}-1}\right)\varphi_{n_{k+1}} n_{k+1}^{1-\alpha} \leqq$$

$$\leqq K_2(\nu)\,\omega^\nu\left(\frac{1}{n_{k+1}}\right)\varphi_{n_{k+1}} n_{k+1}^{1-\alpha}.$$

(1. 38) and (1. 39) imply, by (1. 36), that

$$\sigma_k \leqq K_3(\nu)(\varphi_{n_k} n_k^{1-\alpha} B_{n_k}^\nu + \varphi_{n_{k+1}} n_{k+1}^{1-\alpha} B_{n_{k+1}}^\nu).$$

Hence and from (1. 37), on account of (1. 24), the statements (1. 28) follow.

In order to prove (1. 29) we set $J_j = [2^j, 2^{j+1})$ $(j=0, 1, \ldots)$. If $n_k \in J_{i_k}$, then for any l, $l \neq k$, we have $n_l \notin J_{i_k}$, indeed $n_{k+1} > 2n_k$. Therefore, if $n_k \in J_{i_k}$, then

$$B_{2^{i_k}} - B_{2^{i_k+1}} = B_{n_k} - B_{n_{k+1}}.$$

Using this and $B_{n_{k+1}} \leqq \frac{1}{2} B_{n_k}$ we have

$$\sum_{n=0}^{\infty}\varphi_{2^n} 2^{n(1-\alpha)}(B_{2^n} - B_{2^{n+1}})^\nu \geqq K_1 \sum_{k=1}^{\infty}\varphi_{2^{i_k}} 2^{i_k(1-\alpha)}(B_{n_k} - B_{n_{k+1}})^\nu \geqq$$

$$\geqq K_2 \sum_{k=1}^{\infty}\varphi_{2^{i_k}} 2^{i_k(1-\alpha)} B_{n_k}^\nu \geqq K_3 \sum_{k=1}^{\infty}\varphi_{n_k} n_k^{1-\alpha} B_{n_k}^\nu.$$

Hence, by (1. 28), the statement (1. 29) follows, and this completes the proof.

§ 2. Proof of the theorems

Proof of Theorem 1. Let $F(x)$ be the same function as in Lemma 4. It is well known that for any nonnegative nondecreasing function $\chi(u)$ on $[0, \infty)$

$$\int_0^1 \chi(|f(x)|)\,dx = \int_0^1 \chi(F(x))\,dx$$

(see [7], p. 54). Therefore it is sufficient to prove our statements for $F(x)$.

Put

$$E_n = \{z\colon\ z \in [0, 1],\ n \leqq F(z) < n+1\} \qquad (n=0, 1, \ldots).$$

It is clear that $E_n E_m = 0$ $(n \neq m)$, $\sum_{n=0}^{\infty} E_n = [0, 1]$ and $E_n = \{\alpha_{n+1}, \alpha_n\}$, where $\alpha_n \downarrow 0$ as $n \to \infty$. Set

$$A_n = \int_0^{\alpha_n} F^p(z)\,dz.$$

Then $A_n \to 0$ as $n \to \infty$ and for $n \geqq n_0$

$$1 \geqq A_n = \int_0^{\alpha_n} F^p(z)\,dz \geqq n\alpha_n,$$

that is

(2. 1) $$\alpha_n \leqq 1/n \quad \text{for} \quad n \geqq n_0.$$

If $\nu = p$ then we have to prove that

$$I = \int_0^1 F^p(z)\Phi(F(z))\,dz < \infty.$$

Using Abel transformation we obtain

(2. 2) $$I \leqq \sum_{n=0}^{\infty} \int_{E_n} F^p(z)\,dz \sum_{k=1}^{n} \varphi_k k^{-1} \leqq \sum_{k=1}^{\infty} \varphi_k k^{-1} \sum_{n=k}^{\infty} \int_{E_n} F^p(z)\,dz \leqq \sum_{k=1}^{\infty} \varphi_k k^{-1} A_k.$$

Hence for $p = \nu = 1$, by (2), (1. 3) and (2. 1), the statement (1) obviously follows. If $\nu = p > 1$, using (2), (3), (2. 1), (2. 2) and Lemma 8, we get with $\varepsilon = \varepsilon(p)$ that

$$I \leqq K \sum_{k=1}^{\infty} \varphi_k k^{-1-\varepsilon} \left(\sum_{n=1}^{k} n^{\varepsilon-1} \omega_p^p\left(\frac{1}{n}, f\right) + 1 \right) \leqq$$

$$\leqq K_1 \left\{ \sum_{n=1}^{\infty} n^{\varepsilon-1} \omega_p^p\left(\frac{1}{n}, f\right) \sum_{k=n}^{\infty} \varphi_k k^{-1-\varepsilon} + 1 \right\} \leqq K_2 \left\{ \sum_{n=1}^{\infty} n^{-1} \varphi_n \omega_p^p\left(\frac{1}{n}, f\right) + 1 \right\} < \infty;$$

thus (1) is verified for $\nu = p > 1$.

If $\nu > p$, then

(2. 3) $$I_1 \equiv \int_0^1 F(z)^{\nu - \frac{\nu}{p} + 1} \Phi(F(z))\,dz = \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} F(z)^{\nu - \frac{\nu}{p} + 1} \Phi(F(z))\,dz \leqq$$

$$\leqq \sum_{n=1}^{\infty} 2^{-n} F(2^{-n})^{\nu - \frac{\nu}{p} + 1} \Phi(F(2^{-n})) \equiv \Sigma_1.$$

By (4)

$$\varphi_{N \cdot m} \leqq K(N)\varphi_m; \qquad m = 1, 2, \ldots,$$

and by (1. 4)

$$F(2^{-n}) \leqq K 2^n.$$

According to these we have

$$\Phi(F(2^{-n})) \leqq K_1 \varphi_{2^n} F(2^{-n})^{\frac{\nu}{p} - 1}.$$

Hence

$$\Sigma_1 \leqq K_1 \sum_{n=1}^{\infty} 2^{-n} \varphi_{2^n} F(2^{-n})^{\nu}.$$

Using this and (1. 4) we obtain

(2. 4)
$$\Sigma_1 \leqq K_2 \sum_{n=1}^{\infty} \varphi_{2^n} 2^{-n}\left\{1+\sum_{k=1}^{n} 2^{k/p}\omega_p(2^{-k},f)\right\}^{\nu} \leqq$$
$$\leqq K_3+K_4 \sum_{n=1}^{\infty} \varphi_{2^n} 2^{-n}\left\{\sum_{k=1}^{n} 2^{k/p}\omega_p(2^{-k},f)\right\}^{\nu} \leqq$$
$$\leqq K_5+K_6 \sum_{m=1}^{\infty} \varphi_m m^{-2}\left(\sum_{i=1}^{m} i^{\frac{1}{p}-1}\,\omega_p\left(\frac{1}{i},f\right)\right)^{\nu} = K_5+\Sigma_2.$$

Using (2), (4) and Lemma 1 $(\alpha_k=\varphi_k k^{-2},\ \eta_n \leqq Kn,\ \gamma=\nu)$ we get

(2. 5)
$$\Sigma_2 \leqq K_7 \sum_{m=1}^{\infty} \varphi_m m^{-2} m^{\nu}\left(m^{\frac{1}{p}-1}\,\omega_p\left(\frac{1}{m},f\right)\right)^{\nu} = K_7 \sum_{m=1}^{\infty} \varphi_m m^{\frac{\nu}{p}-2}\,\omega_p^{\nu}\left(\frac{1}{m},f\right) < \infty.$$

Collecting the estimates (2. 3), (2. 4) and (2. 5), we obtain that $J_1<\infty$, that is, (1) is proved.

We have completed our proof.

Proof of Theorem 2. An application of Lemma 4, (5) and (6) now yield

$$\int_0^1 \psi(|f(x)|)\,dx = \int_0^1 \psi(F(x))\,dx = \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \psi(F(x))\,dx \leqq$$
$$\leqq \sum_{n=0}^{\infty} 2^{-n-1}\psi(F(2^{-n-1})) \leqq \sum_{n=0}^{\infty} 2^{-n-1}\psi\left(K\left\{1+\sum_{k=1}^{n} 2^{k/p}\omega_p(2^{-k},f)\right\}\right) \leqq$$
$$\leqq K_1 \sum_{n=0}^{\infty} 2^{-n}\psi\left(1+\sum_{i=1}^{2^n} i^{\frac{1}{p}-1}\,\omega_p\left(\frac{1}{i},f\right)\right) \leqq$$
$$\leqq K_2 \sum_{m=1}^{\infty} (m+1)^{-2}\psi\left(1+\sum_{i=1}^{m} i^{\frac{1}{p}-1}\,\omega_p\left(\frac{1}{i},f\right)\right) \leqq K_3 \sum_{m=1}^{\infty} m^{-2}\psi\left(m^{1/p}\omega_p\left(\frac{1}{m},f\right)\right).$$

Hence, by (7), we obtain the statement of Theorem 2.

Proof of Theorem 3. The sufficiency of (9) has been proved by Theorem 1. The necessity of (9) will be proved indirectly. Suppose that

(2. 6)
$$\sum_{n=1}^{\infty} n^{\frac{\nu}{p}-2}\varphi_n\omega^{\nu}\left(\frac{1}{n}\right) = \infty,$$

yet, (8) holds. According to Lemma 6 we can assume that $\omega(\delta)$ is concave. Now we can construct a function $f_0(x)$ leading to contradiction.

If $\nu=p$, then all the requirements of Lemma 9 are satisfied, thus we have a sequence $\{\bar{B}_n\}$ such that it satisfies (1. 15) and (1. 16) with $\alpha=1$.

If $\nu > p$, using Lemma 10 with $\alpha = 2-\frac{\nu}{p}$, we have sequences $\{\hat{B}_n\}$ and $\{n_k\}$ satisfying (1. 25)—(1. 29) with $\alpha = 2-\frac{\nu}{p}$.

Now define $f_0(x)$ as follows:

$$f_0(x) = \begin{cases} \varrho_n, & \text{if} \quad x = 3\cdot 2^{-n-2}, \\ 0, & \text{if} \quad x = 0,\ x\in[1/2, 1],\ x = 2^{-n}, \\ \text{linear on } [2^{-n-1}, 3\cdot 2^{-n-2}], [3\cdot 2^{-n-2}, 2^{-n}] \end{cases}$$

$(n = 1, 2, \dots)$, where $\varrho_n = 2^{(n+1)\frac{1}{p}}(B_{2^n}^p - B_{2^{n+1}}^p)^{\frac{1}{p}}$ and

$$B_n = \begin{cases} \bar{B}_n, & \text{if} \quad p = \nu, \\ \hat{B}_n, & \text{if} \quad p < \nu. \end{cases}$$

First we show that $f_0(x)\in H_p^{\omega(\delta)}$.

Let

$$h\in(2^{-k-3}, 2^{-k-2}], \quad k \geqq 2. \tag{2. 7}$$

Then

$$\int_0^{1-h} |f_0(t+h)-f_0(t)|^p\,dt = \left(\int_0^{3h} + \int_{3h}^{1-h}\right) |f_0(t+h)-f_0(t)|^p\,dt = I_1+I_2.$$

By (1. 15) and (1. 25) we have

$$I_1 \leqq K(p) \int_0^{4h} |f_0(x)|^p\,dx \leqq K \int_0^{2^{-k}} |f_0(x)|^p\,dx \leqq K \sum_{n=k}^{\infty} \int_{2^{-n-1}}^{2^{-n}} |f_0(x)|^p\,dx \leqq$$

$$\leqq K_1 \sum_{n=k}^{\infty} \varrho_n^p 2^{-n-1} = K_1 \sum_{n=k}^{\infty} (B_{2^n}^p - B_{2^{n+1}}^p) \leqq K_1 B_{2^k}^p \leqq K_2\,\omega^p(h).$$

To estimate J_2 we use the inequalities

$$|f_0(t+h)-f_0(t)| \leqq h2^{n+2}(\varrho_n+\varrho_{n-1}), \quad \text{if} \quad 2^{-n-1} \leqq t \leqq 2^{-n}(1 \leqq n \leqq k-1)$$

and (1. 15). In fact

$$I_2 \leqq \int_{2^{-k}}^{2^{-1}} |f_0(t+h)-f_0(t)|^p\,dt = \sum_{n=1}^{k-1} \int_{2^{-n-1}}^{2^{-n}} |f_0(t+h)-f_0(t)|^p\,dt \leqq$$

$$\leqq K(p)h^p \sum_{n=0}^{k} 2^{n(p-1)} \varrho_n^p \leqq K_1 h^p \sum_{n=0}^{k} 2^{np}(B_{2^n}^p - B_{2^{n+1}}^p) \leqq K_1 h^p \sum_{n=0}^{k} 2^{np} B_{2^n}^p \leqq$$

$$\leqq K_2 h^p \sum_{i=1}^{2^k} i^{p-1} B_i^p \leqq K_3 h^p 2^{kp}\omega^p(2^{-k}) \leqq K_4\,\omega^p(h).$$

Summing up we get

(2. 8) $$f_0(x)\in H_p^\omega.$$

Next we prove that

(2. 9) $$f_0(x)\notin L^{\nu-\frac{\nu}{p}+1}\Phi(L).$$

First we demonstrate (2. 9) if $\nu=p$. In this case, by (2. 6) and (1. 16), we have

(2. 10) $$\sum_{n=1}^{N}\varphi_n n^{-1}B_n^p\to\infty\quad\text{as}\quad N\to\infty.$$

By (1. 13) there exists a number K_1 such that $\varphi_{2n}\leqq K_1\varphi_n$ for all n, and since $B_n\downarrow 0$ for any N there exists an integer N_1 such that $B_{N1}<\frac{1}{2}B_N$. Using these facts and (2. 10), an easy computation gives that

(2. 11) $$\sum_{n=1}^{\mu}\Phi(2^n)\varrho_n^p 2^{-n}\to\infty\quad\text{as}\quad\mu\to\infty.$$

Indeed, if $2^\mu>N_1$, we have

$$\sum_{k=1}^{N}\varphi_k k^{-1}B_k^p\leqq 2\sum_{k=1}^{N}\varphi_k k^{-1}(B_k^p-B_{N_1}^p)\leqq 2\sum_{k=1}^{2^\mu}\varphi_k k^{-1}(B_k^p-B_{2^\mu}^p)\leqq$$

$$\leqq 2\left(\sum_{n=1}^{\mu}\sum_{k=2^{n-1}+1}^{2^n}\varphi_k k^{-1}B_k-\Phi(2^\mu)B_{2^\mu}^p\right)+K_2\leqq$$

$$\leqq 2\left(\sum_{n=1}^{\mu}B_{2^{n-1}}^p\sum_{k=2^{n-1}+1}^{2^n}\varphi_k k^{-1}-\Phi(2^\mu)B_{2^\mu}^p\right)\leqq$$

$$\leqq K_3\left(\sum_{n=1}^{\mu}B_{2^{n-1}}^p\sum_{k=2^{n-2}+1}^{2^{n-1}}\varphi_k k^{-1}-\Phi(2^\mu)B_{2^\mu}^p\right)+K_2\leqq$$

$$\leqq K_3\left(\sum_{i=1}^{\mu-1}B_{2^i}^p(\Phi(2^i)-\Phi(2^{i-1}))-\Phi(2^\mu)B_{2^\mu}^p\right)+K_4\leqq$$

$$\leqq K_3\sum_{n=1}^{\mu-1}\Phi(2^n)(B_{2^n}^p-B_{2^{n+1}}^p)+K_4\leqq K_3\sum_{n=1}^{\mu}\Phi(2^n)\varrho_n^p 2^{-n-1}+K_4,$$

which proves (2. 11) by (2. 10).

It is clear that

$$\int_0^1|f_0(x)|^p\Phi\left(\frac{1}{x}\right)dx=\sum_{n=0}^{\infty}\int_{2^{-n-1}}^{2^{-n}}|f_0(x)|^p\Phi\left(\frac{1}{x}\right)dx\geqq$$

$$\geqq\sum_{n=0}^{\infty}\Phi(2^n)\int_{2^{-n-1}}^{2^{-n}}|f_0(x)|^p\,dx\geqq K_5\sum_{n=0}^{\infty}\Phi(2^n)\varrho_n^p 2^{-n},$$

and thus, by (2. 11), we have

$$\int_0^1 |f_0(x)|^p \Phi\left(\frac{1}{x}\right) dx = \infty. \tag{2. 12}$$

Since $\varphi_{k^2} \leqq K_1 \varphi_k$, we have

$$\Phi(u^2) \leqq K\Phi(u),$$

thus by (2. 12), applying Lemma 3, we obtain

$$\int_0^1 |f_0(x)|^p \Phi(|f_0(x)|)\, dx = \infty,$$

that is,

$$f_0(x) \notin L^p\Phi(L). \tag{2. 13}$$

Hereby (2. 9) is proved for $p=\nu$.

Next let us suppose that $\nu>p$. By $\varphi_{k^2} \leqq K_1 \varphi_k$ we have

$$\Phi(x) = \sum_{k=1}^{x} k^{\frac{\nu}{p}-2} \varphi_k \geqq \sum_{x/2}^{x} k^{\frac{\nu}{p}-2} \varphi_k \geqq K_1 \varphi_{[x]} x^{\frac{\nu}{p}-1}.$$

Using this we get

$$\int_0^1 |f_0(x)|^{\nu-\frac{\nu}{p}+1} \Phi(|f_0(x)|)\, dx \geqq K_2 \int_0^1 |f_0(x)|^\nu \varphi(|f_0(x)|)\, dx, \tag{2. 14}$$

where

$$\varphi(x) = \begin{cases} 0, & \text{if } x \in [0, 1), \\ \varphi_n, & \text{if } x = n, \\ \text{linear between } n \text{ and } n+1 \end{cases}$$

($n=1, 2, \ldots$). Furthermore, an application of Lemma 10 with $\alpha = 2-\frac{\nu}{p}$, especially (1. 29), gives that

$$\sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\left(\frac{\nu}{p}-1\right)} (B_{2^n} - B_{2^{n+1}})^\nu = \infty$$

and this implies

$$J \equiv \int_0^1 |f_0(x)|^\nu \varphi\left(\frac{1}{x}\right) dx = \infty. \tag{2. 15}$$

Indeed, we have

$$J = \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} |f_0(x)|^\nu \varphi\left(\frac{1}{x}\right) dx \geqq K_1 \sum_{n=1}^{\infty} \varphi_{2^n} \varrho_n^\nu 2^{-n} \geqq$$

$$\geqq K_2 \sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\left(\frac{\nu}{p}-1\right)} (B_{2^n}^p - B_{2^{n+1}}^p)^{\frac{\nu}{p}} \geqq K_2 \sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\left(\frac{\nu}{p}-1\right)} (B_{2^n} - B_{2^{n+1}})^\nu.$$

Applying Lemma 3, by (2. 15) and (2. 14), we obtain (2. 9) also for $\nu>p$.

The statements (2. 8) and (2. 9), however, contradict (2. 6) and hereby the necessity of (9) is proved.

The proof of Theorem 3 is thus completed.

References

[1] А. В. Ефимов, Линейные методы приближения непрерывных периодических функций, *Матем. сб.*, **54**, (1961), 51—90.

[2] L. Leindler, Über verschiedene Konvergenzarten trigonometrischer Reihen. III, *Acta Sci. Math.*, **27** (1966), 205—215.

[3] H. P. Mulholland, Concerning the generalization of the Young—Hausdorff theorem, *Proc. London Math. Soc.*, **35** (1933), 257—293.

[4] П. Л. Ульянов, О вложении некоторых классов функций, *Матем. заметки*, **1** (1967), 405—414.

[5] ———— Вложение некоторых классов функций H_p^ω, *Изв. АН СССР*, сер. матем., **32** (1968), 649—686.

[6] ———— Теоремы вложения и наилучшие приближения, *Докл. АН СССР*, **184** (1968), 1044—1047.

[7] А. Зигмунд, *Тригонометрические ряды*. I (Москва, 1965).

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